The mathematics of bell ringing

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Outline

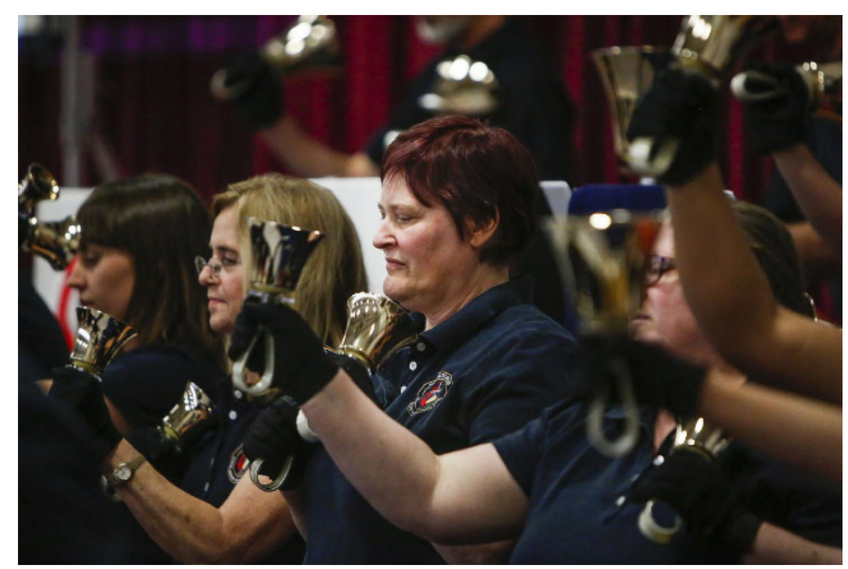
- Introduction to bell ringing and change ringing
- Specific types of change ringing sequences
- Mathematics of change ringing
    - Steinhaus–Johnson–Trotter algorithm
    - Cayley graphs

Introduction

Why bells?

- I have been ringing handbells for almost 20 years

- I currently:
  - ring with an adult choir,
  - direct a middle school choir,
  - assist in directing a high school choir.



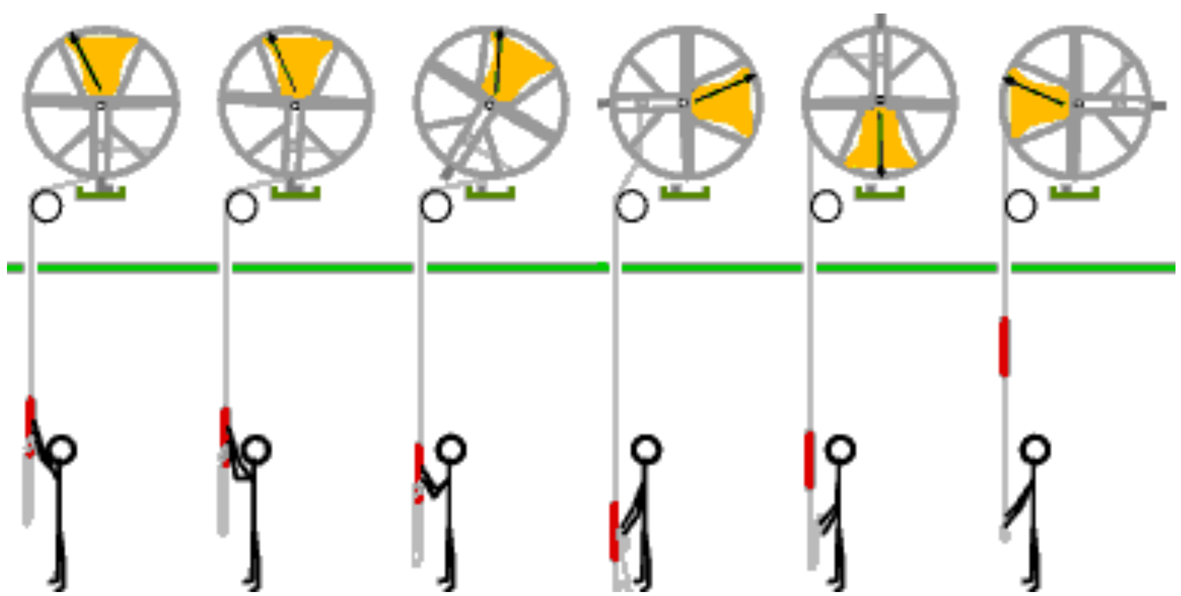
History of handbells

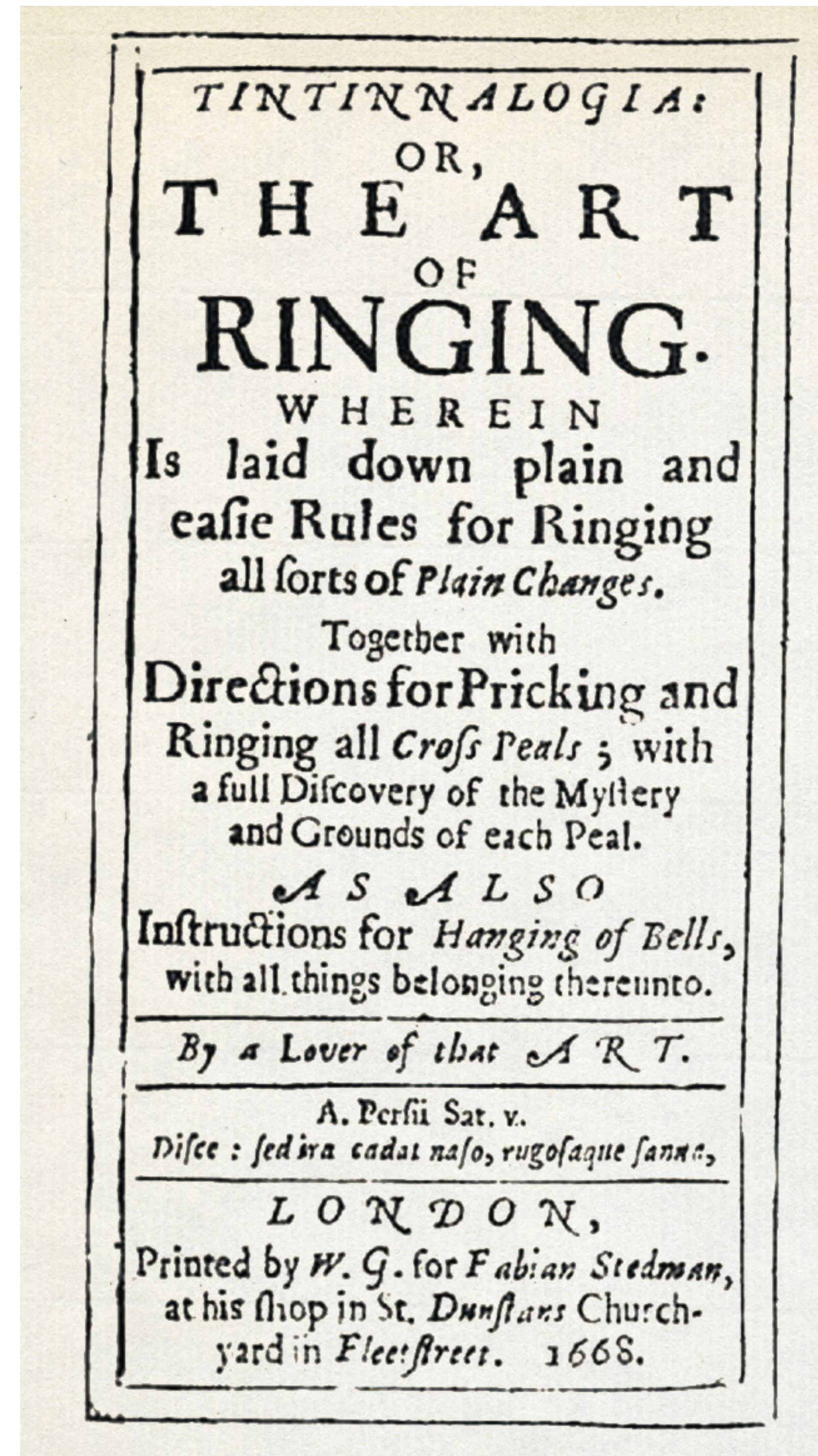
- Handbells originated in 1690s in England for **change ringers** to practice peals outside the towers
- Came to the United States in 1901, music specifically arranged for handbell choirs around 1960s
- Handbell choirs are measured by octave ranges and each player rings "2" notes

The Wesley Bell Ringers, Salt Lake City UT

History of change ringing

- English full-circle tower bells were invented in early 1600s
- Ringing the bells, as opposed to chiming, is called "change ringing"





TINTINNALOGIA:
OR,
THE ART
OF
RINGING.
WHEREIN
Is laid down plain and easie Rules for Ringing all sorts of *Plain Changes*.
Together with
Directions for Pricking and Ringing all *Cross Peals*; with a full Discovery of the Mystery and Grounds of each Peal.
AS ALSO
Instructions for *Hanging of Bells*, with all things belonging thereunto.

By a Lover of that ART.

A. Persii Sat. v.
Disce: sed ira cadat naso, rugosaque sanna,

LONDON,
Printed by *W. G.* for *Fabian Stedman*, at his shop in St. *Dunstans* Church-yard in *Fleetstreet*. 1668.

Change ringing

Definition: An **extent**, or a full peal, is the ability to ring a tower's bells in every possible order

Example: Suppose your bell tower has 3 bells

1	2	3
3	2	1
2	1	3
3	1	2
1	3	2
2	3	1

An extent would involve 6 sequences! $3 \times 2 \times 1 = 6$

Change ringing

- **What if your bell tower has 4 bells?**

$4! = 4 \times 3 \times 2 \times 1 = 24$ sequences (around 30 sec.)

- **What if your bell tower has 6 bells?**

$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ sequences (about 25 min.)

- **What if your bell tower has 8 bells?**

$8! = 40320$ sequences (18 hours in England, 1963)

- **For n bells, the sequences are elements of the symmetric group, S_n**

Change ringing nomenclature

n	Name	$n!$
3	Singles	6
4	Minimus	24
5	Doubles	120
6	Minor	720
7	Triples	5,040
8	Major	40,320
9	Caters	362,880
10	Royal	3,628,800
11	Cliques	39,916,800
12	Maximus	479,001,600

Change Ringing Technique: Plain change

Definitions of change ringing

- Physical constraints:
  - You can only swap neighboring bells
  - No repeating sequences
  - Want to start and end with bells in highest to lowest order
- **Definition:** A **true extent** is an extent with no repeated sequences
- **Definition:** A **round** is a sequence of bells in highest to lowest order (the "identity element")
- **Definition:** A **plain change** is a change ringing technique where one bell is swapped with its neighbor

Ring a true extent on 3 bells

- **Example:** Let's go back to our tower of three bells

  Using cycle notation and Cauchy's two line notation, we define plain changes as

$$a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (1\,2)(3) = (1\,2)$$

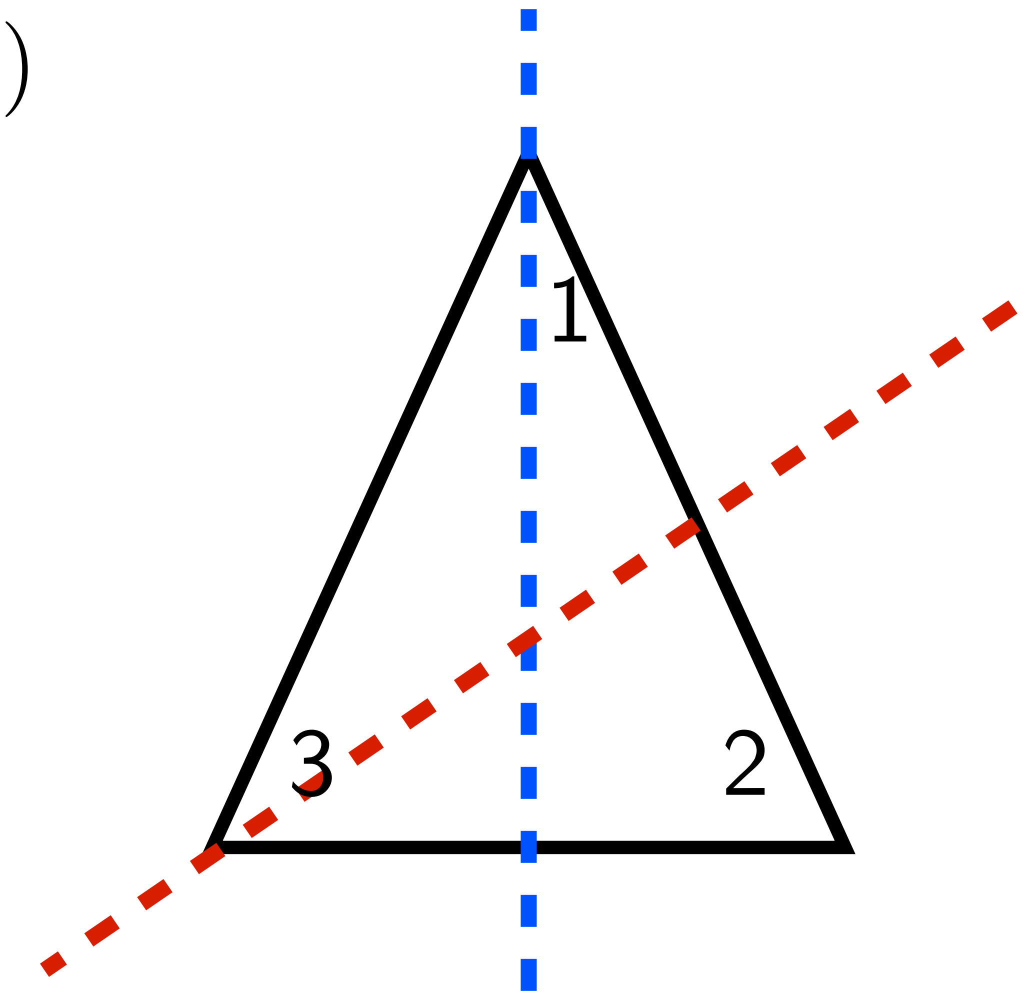
$$b = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (1)(2\,3) = (2\,3)$$

Ring a true extent on 3 bells

- **Example:** Let's go back to our tower of three bells. Using only plain changes,

$$a = (1\,2)$$
$$b = (2\,3)$$

	1	2	3
a	2	1	3
b	2	3	1
a	3	2	1
b	3	1	2
a	1	3	2
b	1	2	3



Triangle denoting positions

Change ringing on 4 bells

- **Example:** What if we had 4 bells in our tower and rang only plain changes?

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} = (1)(2)(3\,4) = (3\,4)$$

$$b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} = (1)(2\,3)(4) = (2\,3)$$

$$c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} = (1\,2)(3)(4) = (1\,2)$$

Plain changes on 4 bells (Minimus)

	1	2	3	4	No.
a	1	2	4	3	1
b	1	4	2	3	2
c	4	1	2	3	3
a	4	1	3	2	4
c	1	4	3	2	5
b	1	3	4	2	6
a	1	3	2	4	7
c	3	1	2	4	8
a	3	1	4	2	9
b	3	4	1	2	10
c	4	3	1	2	11

	4	3	1	2	
a	4	3	2	1	12
c	3	4	2	1	13
b	3	2	4	1	14
a	3	2	1	4	15
c	2	3	1	4	16
a	2	3	4	1	17
b	2	4	3	1	18
c	4	2	3	1	19
a	4	2	1	3	20
c	2	4	1	3	21
b	2	1	4	3	22
a	2	1	3	4	23
c	**1**	**2**	**3**	**4**	24

Ring a true extent with 4 bells, beginning and ending in rounds!

Mathematics and plain changing

Steinhaus–Johnson–Trotter algorithm

- In 1963, was published to generate all permutations of n elements
- Recursive algorithm: Sequence of permutations for n can be formed from sequence of permutations for $n - 1$ by placing n into each possible position
- If permutation on $n - 1$ is even, then n is placed in descending order from n to 1
- Else, n is placed in ascending order from 1 to n

SJT Algorithm example

- Example: SJT with 4 elements

1. Start with even and odd permutations of 3 elements

Even (even # of swaps)	Odd (odd # of swaps)
2 3 1	**1 3 2**
3 1 2	2 1 3
1 2 3	3 2 1

SJT Algorithm example

- Example: SJT with 4 elements

2. Place 4 in descending order for even permutations, ascending order for odd permutations

2 3 1		2 3 1 **4**	**4** 1 3 2		**1 3 2**
3 1 2	→	2 3 **4** 1	1 **4** 3 2	←	2 1 3
1 2 3		2 **4** 3 1	1 3 **4** 2		3 2 1
		4 2 3 1	1 3 2 **4**		
Even (even # of swaps)					Odd (odd # of swaps)

SJT Algorithm example:

- Example: SJT with 4 elements

2. Place 4 in descending order for even permutations, ascending order for odd permutations

This enumerates plain changes!

Even				Odd
2 3 1		2 3 1 **4**	**4** 1 3 2		**1 3 2**
3 1 2	→	2 3 **4** 1	1 **4** 3 2	←	2 1 3
1 2 3		2 **4** 3 1	1 3 **4** 2		3 2 1
		4 2 3 1	1 3 2 **4**		

Even (even # of swaps)

Odd (odd # of swaps)

SJT Algorithm example: 4 elements

Even	1 2 3 **4**	Odd	3 2 1 **4**
	1 2 **4** 3		3 2 **4** **1**
	1 **4** 2 3		3 **4** 2 1
	4 1 2 3		**4** 3 2 1
Odd	**4** 1 3 2	Even	**4** 2 3 1
	1 **4** 3 2		2 **4** 3 1
	1 3 **4** 2		2 3 **4** 1
	1 3 2 **4**		2 3 1 **4**
Even	3 1 2 **4**	Odd	2 1 3 **4**
	3 1 **4** 2		2 1 **4** 3
	3 **4** 1 2		2 **4** 1 3
	4 3 1 2		**4** 2 1 3

Other change ringing techniques

More moves!

- Now, let's allow multiple swaps to in one move
- **Definition:** A **cross-change** involves swapping multiple bells in one move

$$c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = (1\,2)(3\,4)$$

- **Definition:** A **plain hunt** is a sequence of changes involving a cross-change then a plain change

$$b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} = (1)(2\,3)(4) = (2\,3)$$

Plain hunt on four

$$c = (1\,2)(3\,4)$$
$$b = (2\,3)$$

					No.
Start	1	2	3	4	1
c	2	1	4	3	2
b	2	4	1	3	3
c	4	2	3	1	4
b	4	3	2	1	5
c	3	4	1	2	6
b	3	1	4	2	7
c	1	3	2	4	8
b	1	2	3	4	9

Begins and ends in rounds, but is not an extent

Plain hunt and group theory

- **Definition:** A **group** is a set with an operation $\cdot$ that can combine elements in the group to form another element and satisfy
  1. Closure: $\forall a, b \in G, a \cdot b \in G$
  2. Associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
  3. Identity element: $\exists e \in G \text{ s.t. } a \cdot e = e \cdot a = a$
  4. Inverse element: $\forall a, \exists b = a^{-1} \text{ s.t. } a \cdot b = b \cdot a = 1$

- Moves in plain hunt correspond to the Dihedral group of four elements, D_4, the set of symmetries (rotations, reflections) of a square

Dihedral group, D_4

Rotations

(a)

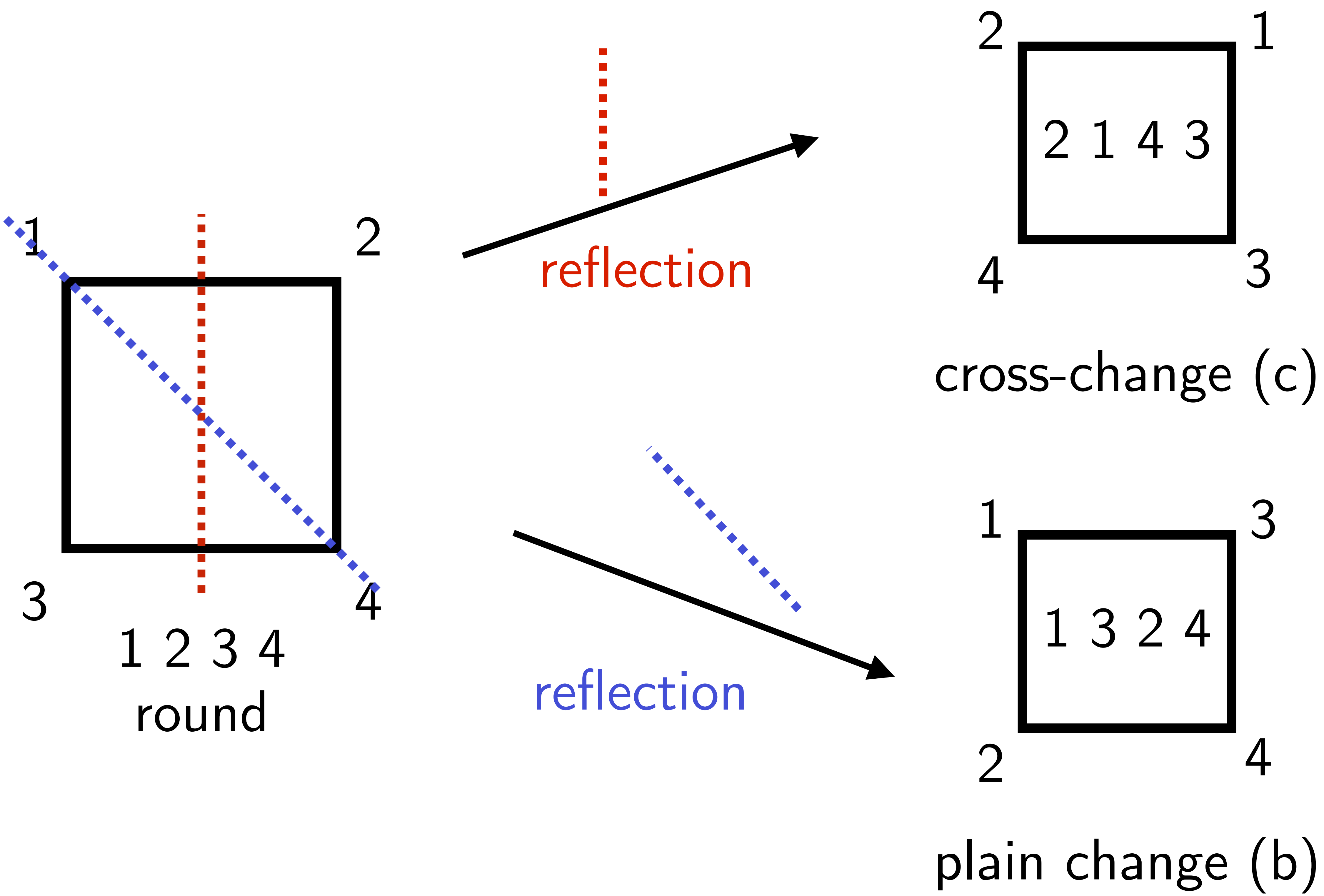
e (identity) a a^2 a^3

Reflections

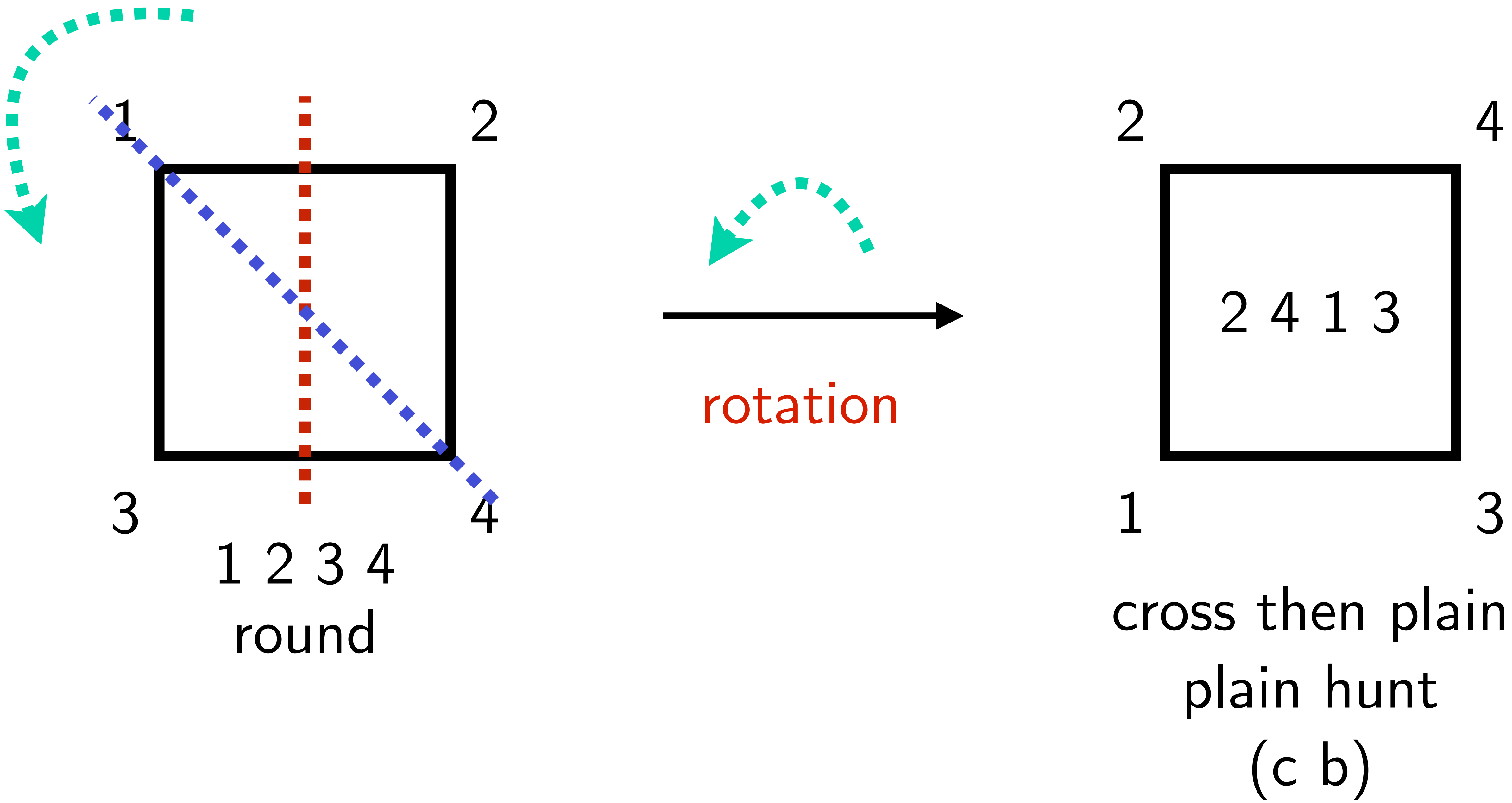
(b)

b ab (ba^3) a^2b(ba^2) a^3b(ba)

Dihedral group, D_4



Dihedral group, D_4



- **Can we extend the plain hunt on four to an extent?**

Plain hunt on four, edited

					No.
Start	1	2	3	4	1
c	2	1	4	3	2
b	2	4	1	3	3
c	4	2	3	1	4
b	4	3	2	1	5
c	3	4	1	2	6
b	3	1	4	2	7
c	1	3	2	4	8
a	1	3	4	2	8

Rather than plain change the middle position, plain change the last two!

$$c = (1\,2)(3\,4)$$

$$b = (2\,3)$$

$$a = (3\,4)$$

The **Plain Bob Minimus** is generated by $\Delta = \{a, b, c\}$

Plain Bob Minimus

	1	2	3	4	1
c	2	1	4	3	2
b	2	4	1	3	3
c	4	2	3	1	4
b	4	3	2	1	5
c	3	4	1	2	6
b	3	1	4	2	7
c	1	3	2	4	8

a	1	3	4	2	9
c	3	1	2	4	10
b	3	2	1	4	11
c	2	3	4	1	12
b	2	4	3	1	13
c	4	2	1	3	14
b	4	1	2	3	15
c	1	4	3	2	16

a	1	4	2	3	17
c	4	1	3	2	18
b	4	3	1	2	19
c	3	4	2	1	20
b	3	2	4	1	21
c	2	3	1	4	22
b	2	1	3	4	23
c	1	2	4	3	24
a	1	2	3	4	

Transition sequence: $(cb)^3ca = (2\,4\,3)$

Other changes

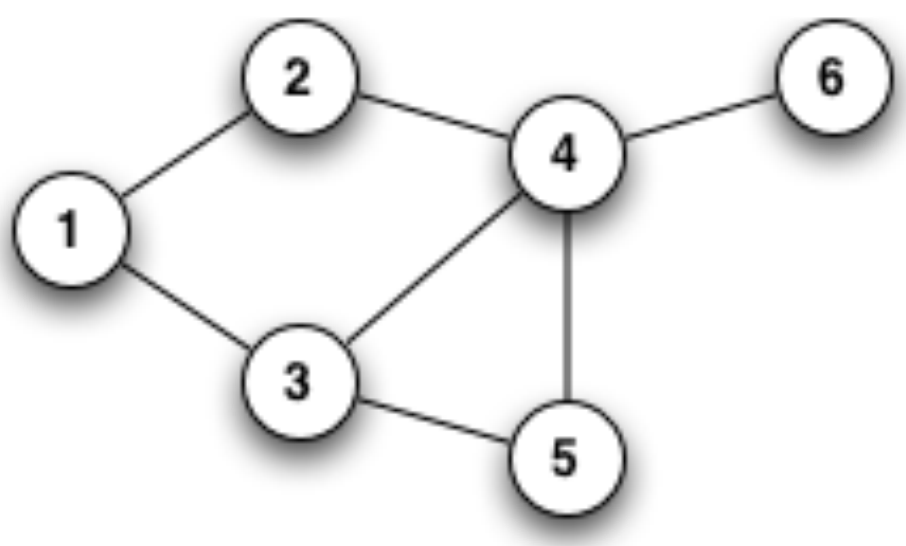
Name of method	Transition sequence
Plain Bob	$(cb)^3ca$
Reverse Bob	$cbcd(cb)^2$
Double Bob	$cbcdcbca$
Canterbury	$cbadabcb$
Reverse Canterbury	$db(cb)^2da$
Double Canterbury	$dbadabda$
Single Court	$db(cb)^2db$
Reverse Court	$cb(ab)^2cb$
Double Court	$db(ab)^2db$
St. Nicholas	$dbcdcbda$
Reverse St. Nicholas	$cbadabca$

$$c = (1\,2)(3\,4)$$
$$b = (2\,3)$$
$$a = (3\,4)$$
$$d = (1\,2)$$

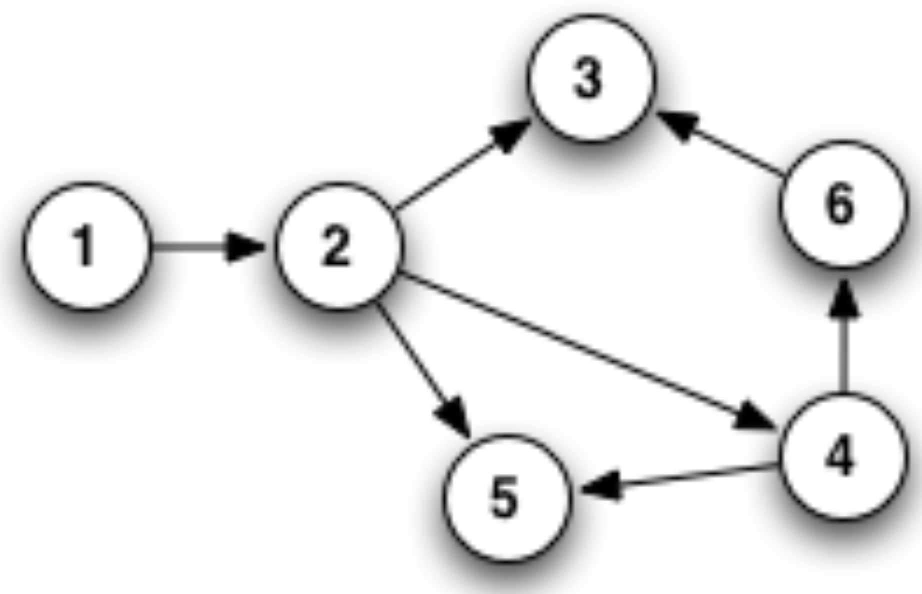
Change ringing and graph theory

Change ringing and graph theory

- We can also look at true extents using graph theory!
- **Definition:** A **graph** is a set of vertices, edges, and a function that defines an ordered pair of vertices to an edge



Undirected

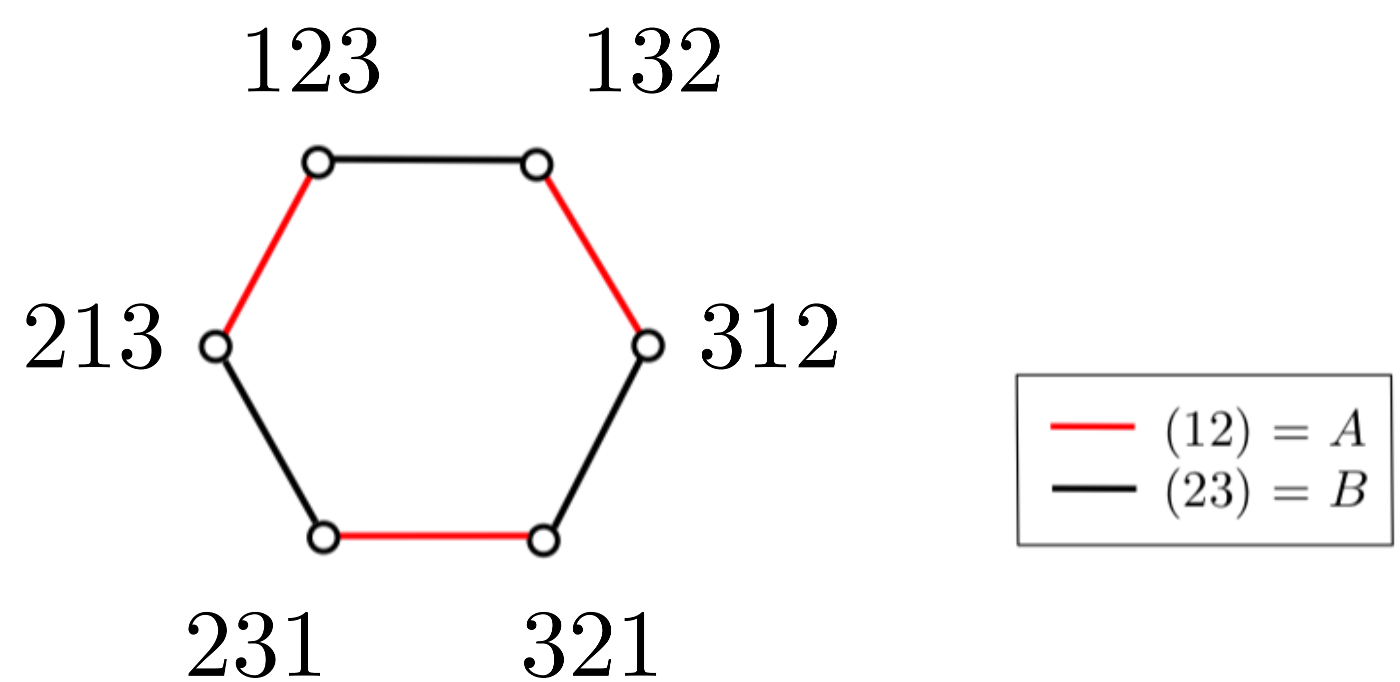


Directed

Change ringing and graph theory

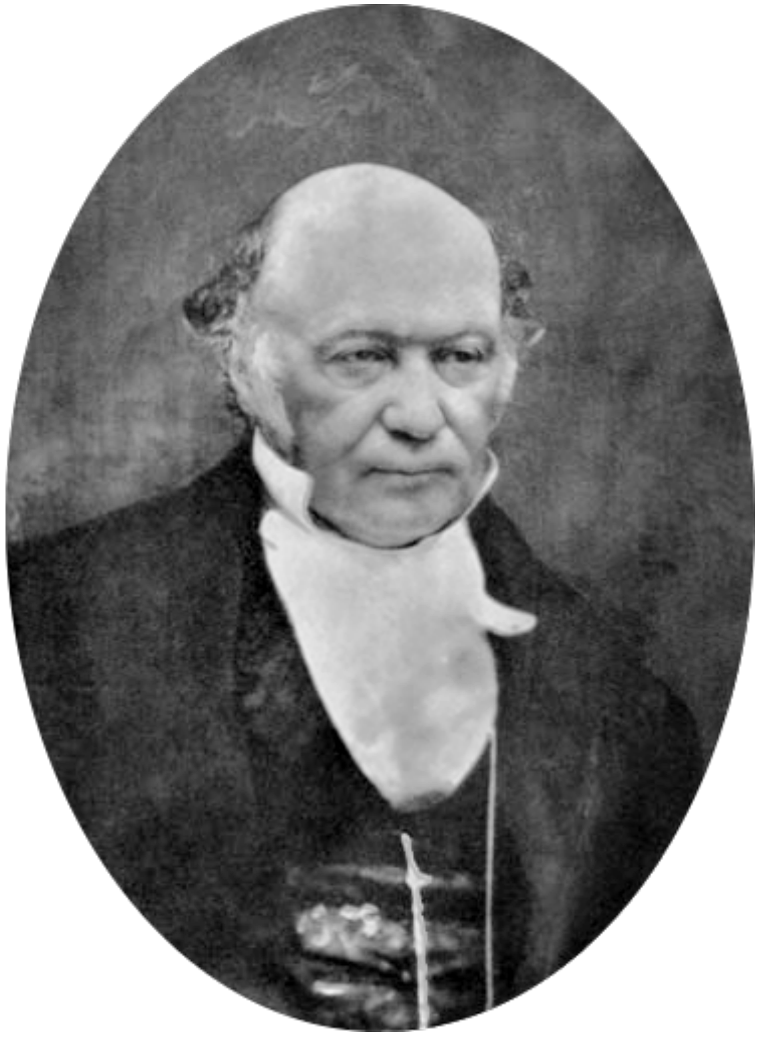
- **Definition:** A **Cayley color graph of a group G** with respect to a generating set (set of moves) is a colored directed graph where
  1. Every vertex is an element in G
  2. Every move is assigned to a color
  3. Edges connect vertices attainable from moves in generating set

Example: Cayley color graph for 3–bell with plain change



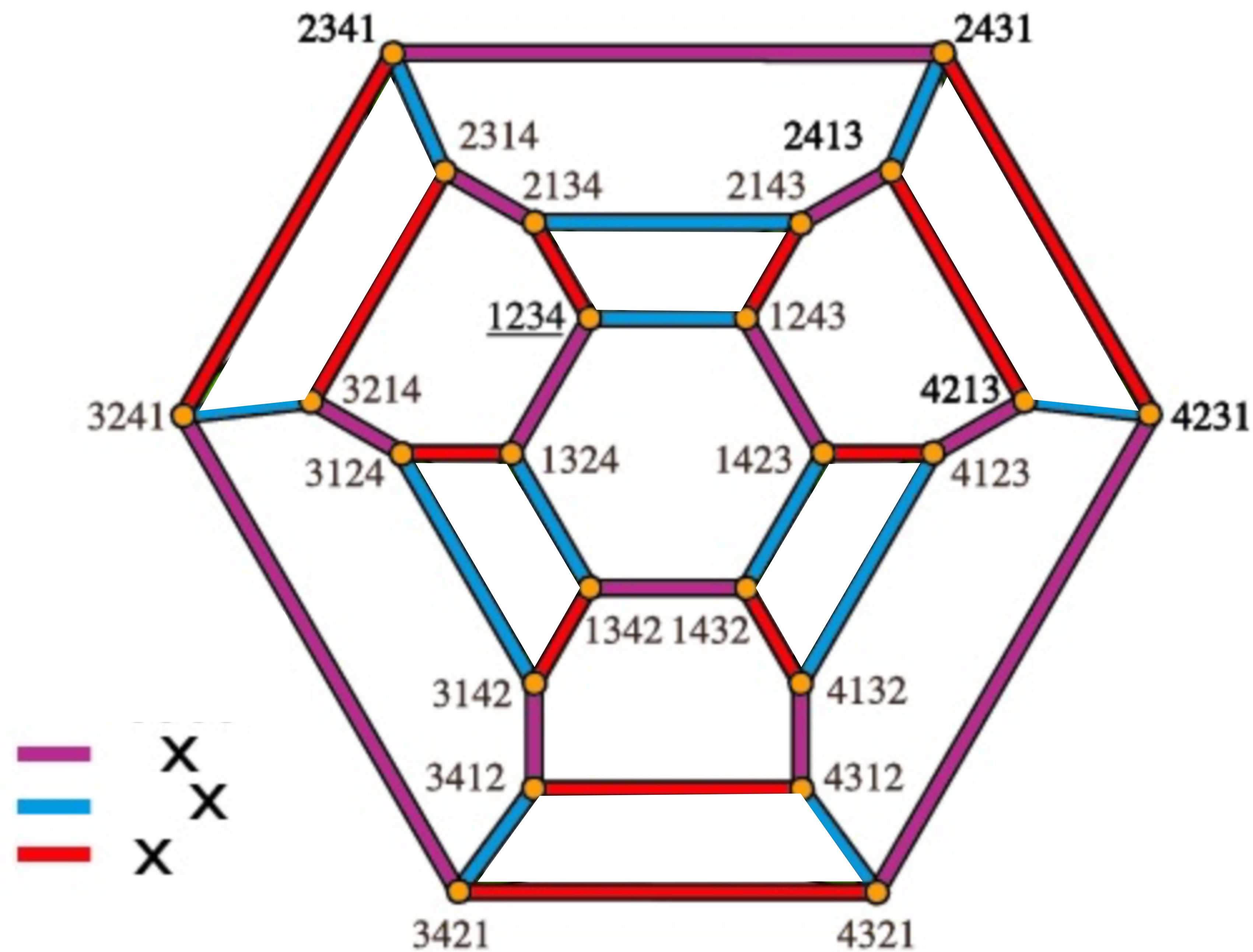
Change ringing and graph theory

- **Definition**: A **Hamiltonian cycle** is a graph that visits every vertex exactly once and returns to the original vertex
  - True extents are Hamiltonian cycles!
- **Theorem:** Let S_n be the set of bell permutations with n bells. An n – bell extent, fulfilling change ringing requirements and using given transitions, can be rung **if and only if** the Cayley color graph of S_n is Hamiltonian



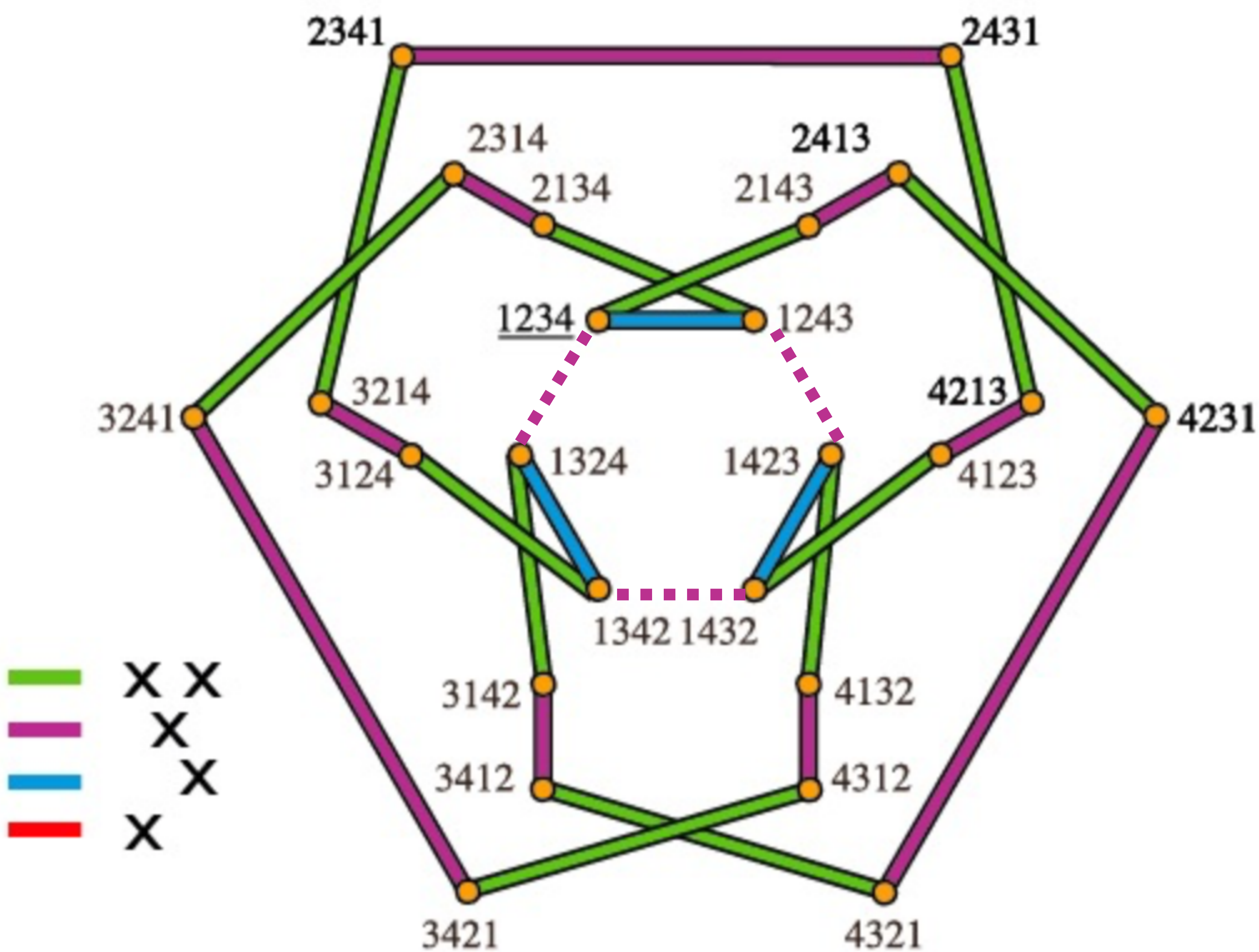
Hamilton

Cayley color graph examples



Example: Cayley color graph for 4–bell with plain changes

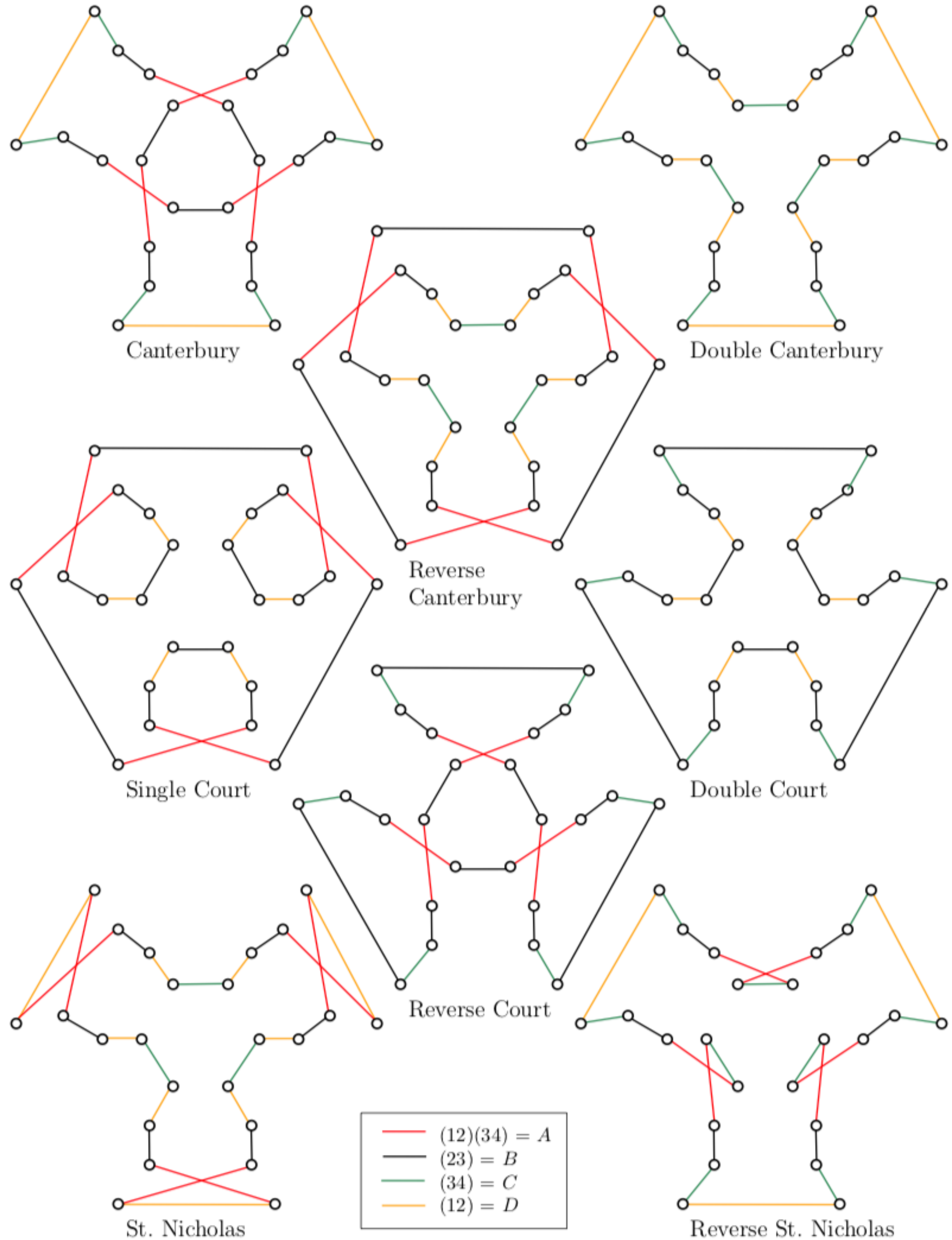
Cayley color graph examples



	1	2	3	4	1
c	2	1	4	3	2
b	2	4	1	3	3
c	4	2	3	1	4
b	4	3	2	1	5
c	3	4	1	2	6
b	3	1	4	2	7
c	1	3	2	4	8

Example: Cayley color graph for Plain Bob Minimus

Even more changes!



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