NAME: MATH 316 – HW 1 Due: Tuesday Aug 25, 2026

HOMEWORK DAY 1 – *First order ODEs $dx/dt = f(t,x)$ §1.1, 1.2*

1. Find the general solution to the following differential equations. These problems include integration by parts, partial fraction decomposition, and the arctangent, all of which we will use throughout the course.

   (a) $\dfrac{dx}{dt} = te^{-2t}$

   (b) $\dfrac{dx}{dt} = \dfrac{1}{t\ln t}$

   (c) $\sqrt{t}\dfrac{dx}{dt} = \cos\sqrt{t}$

   (d) $(1+t^2)\dfrac{dx}{dt} = 1$

(e) $t(1-t)\dfrac{dx}{dt} = 2t + 1$

2. Use the chain rule and the Fundamental Theorem of Calculus to compute the derivative of the following functions

   (a) $f(t) = \displaystyle\int_0^{\sin t} e^{-s^2}\, ds$

   (b) $f(t) = t \displaystyle\int_1^{t} e^{-\sqrt{s}}\, ds$

3. Let $y(t) = e^{-t^2} \int_0^t e^{s^2}\, ds$.

   (a) Find $y'(t) =$

   (b) Show that $y(t)$ satisfies the $y' = 1 - 2ty$.

4. Write the solution to the IVP
$$\frac{dx}{dt} = \frac{e^{-t}}{\sqrt{t}}\ , \quad x(1) = 2$$
as a definite integrals with variable upper limit.

5. Write the solution to the IVP
$$\frac{dP}{dt} = \frac{2}{\sqrt{\pi}} e^{-t^2}\ , \quad P(0) = 0$$
as a definite integrals with variable upper limit.

   Based on the ODE, is $P(t)$ always increasing? always decreasing? neither?

   Given the fact that $\int_0^\infty e^{-s^2}\, ds = \frac{\sqrt{\pi}}{2}$, find the limit $\lim_{t\to\infty} P(t)$=.

   Using the above two facts, can can you sketch a graph of $P(t)$?

6. (Physics) A ball of mass m is tossed upward from a building at height h with initial velocity v_0. Suppose the only force acting on the ball is the force of gravity, $F = -mg$, where g is the acceleration due to gravity. Let $x = x(t)$ denote the height above ground level at time t. By Newton's second law the initial value problem is

$$m\frac{d^2x}{dt^2} = -mg, x(0) = h, x'(0) = v_0.$$

Let $v(t) = dx/dt$ be the balls velocity at time t. Write the 2nd order IVP for x as a 1st order IVP for v. Find the solution of the IVP.

7. State the general solution to the following ODEs. If an initial condition is given, solve the IVPs. (These problems require no or minimal work)

   (a) $dx/dt = 2x$

   (b) $dI/dt = -rI$, $I(0) = I_0$

   (c) $dy/dx = -y$, $y(10) = 5$

   (d) $dP/dt = kP$, $P(1) = 10$

8. In deep water, the intensity of light $I = I(x)$ at a depth x meters below the water surface is modeled by the equation $dI/dx = -1.4I$. At what depth is the light intensity 1% that at the surface? (Observe that the intensity at the surface is $I(0)$.)

9. (Radioactivity) The amount $S(t)$ of radioactive cobalt after a nuclear reactor accident decays exponentially, $S' = -rS$ with solution $S(t) = S(0)e^{-rt}$, where $t = 0$ is the time of the accident, and t is in years. The half life of this material is 5.3 years. That is, $S(5.3) = S(0)/2$.

   (a) Find the value of r (Show all work!)

   (b) The amount $S(0)$ present right after the accident was 100 times the amount acceptable for habitation. How long will it be before the region is habitable?

HOMEWORK DAY 2 – *Show given fcn solves DE. Direction fields. §1.1.1, 1.1.3*

10. Show a given function is a solution

    (a) Show that $x(t) = t^2$ solves the differential equation $\frac{dx}{dt} = \frac{2x}{t}$

    (b) Show that $x(t) = \sqrt{6 - t^2}$ solves the differential equation $\frac{dx}{dt} = -\frac{t}{x}$

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12. Show that the initial value problem $x' = 1 - x^2$, $x(0) = 0$, has the solution $x(t) = \dfrac{e^{2t} - 1}{e^{2t} + 1}$.

13. Consider the ODE $dx/dt = e^{-x}$.

(a) How does the slope of all solution curves $x(t)$ behave

as $x \to \infty$?

As $x \to -\infty$?

(b) Show that $x(t) = \ln(t + C)$ is a one-parameter family of solutions, where C is an arbitrary constant. Plot the integral curves using the values of C given by $C = -2, -1, 0, 1, 2$. On the plot indicate the particular solution that satisfies $x(0) = 0$.

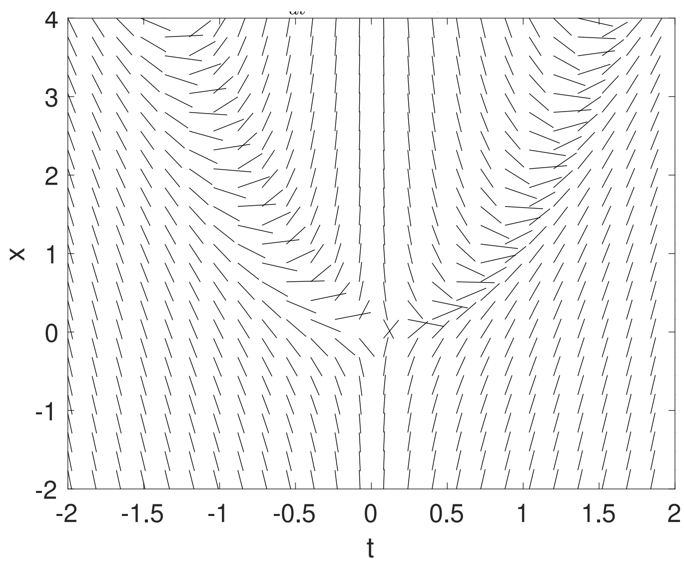
(c) Do the curves $x(t)$ satisfy the behaviour you described in (a)?

14. Consider the differential equation $\dfrac{dx}{dt} + 2x = t^2 + 4t + 7$.
Find a function of the form $x(t) = at^2 + bt + c$ that solves the equation.

15. The direction field for

$$\frac{dx}{dt} = 4t - \frac{2x}{t} = f(t, x)$$

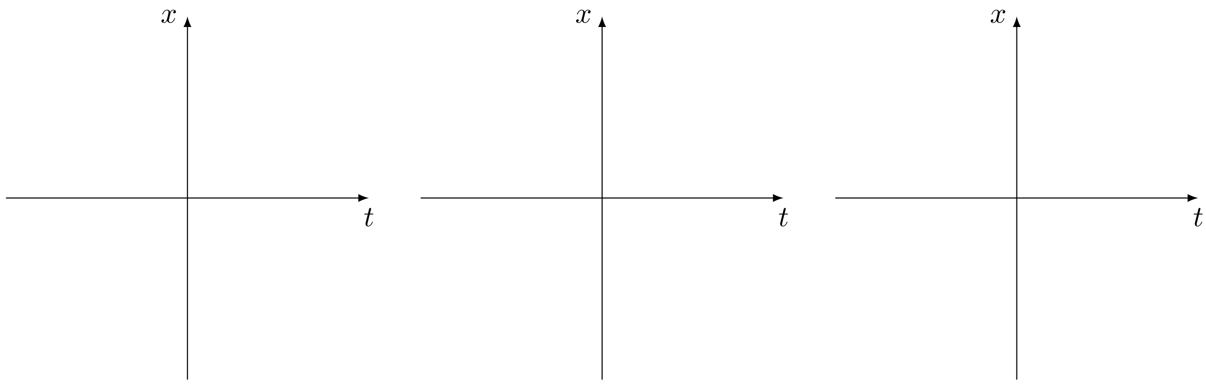
is shown below.



(a) For all points given in the table: (i) compute the slope dx/dt of solutions $x(t)$ going through that point. Highlight the vector with that slope at that point in the direction field. Does the slope of that vector agree (roughly) with the one you computed?

point (t, x)	$(-1, 3)$	$(-1, 2)$	$(-1, 0)$	$(-1, -1)$
slope $f(t, x)$				

(b) One can show that $x(t) = t^2 + \frac{C}{t^2}$ is a family of solutions to the ODE. Sketch the graphs of $x(t)$ for $C = 0, 1, -1$ below, using the method of graphing by superposition we went over in class. Include the point on the graph with $t = 1$.



$C = 0$ $\quad$ $C = 1$ $\quad$ $C = -1$

(c) Add solution curves (integral curves) to the vector field corresponding to $C = 0, 1, -1$. Note that these curves have to be tangent to the vector field and go through the point $(1, x(1))$ that you found above.

16. Find the general solution to the following DEs

    (a) $y' = -2ty \ , \quad y \neq 0$

    (b) $x' = -\frac{x}{t} \ , \quad x \neq 0, \ \ t > 0$

(c) $(60-t)x' = 2x\ ,\quad x \neq 0,\ \ t > 60$

(d) $u' = 5 - 2u\ ,\quad u \neq \frac{5}{2}$

(e) $t\mu' = (1+t)\mu$, $\mu > 0$, $t > 0$

(f) $y' + y + \frac{1}{y} = 0$, $y \neq 0$

17. Solve the given IVPs

(a) $\dfrac{dx}{dt} = tx^3 \; , \quad x(0) = 1$

(b) $\dfrac{dx}{dt} = tx^3 \; , \quad x(0) = 0$

(c) $\dfrac{dy}{dx} = k - ry \; , \quad y(0) = \dfrac{k}{2r}$ where k, r constants

(d) $\dfrac{dy}{dt} = 0.2\left(1 - \dfrac{y}{5}\right) y \ , \quad y(0) = 10$