HOMEWORK DAY 7 – *Population models. §1.5.1*

1. A population $P(t)$ of field mice, where P is the number of mice and t is measured in months from some starting point, grows at the rate of 50% per month ($r = 0.5$ per month). However, owls in the neighbourhood eat them at the rate of 15 mice/day (assume that there are 30 days/month).

   (a) Write down a differential equation for $P(t)$.

   (b) Sketch a plot of P' vs P and draw the phaseline.

   (c) Describe how the evolution of the mouse population in time depends on the initial population $P_0 = P(0)$.

(d) Find the time at which the population becomes extinct if $P_0 = 850$.

2. page 63: 1.

3. page 63: 4.

4. page 66: 10. (Hint: Use $S + I = N$ to obtain an equation for $I'(t)$. Plot the phase line.)

5. page 66: 11. (Hint: Use $S + I = N$ to obtain an equation for $I'(t)$)

6. page 76: 2.

7. page 76: 3. Sketch the region of all acceptable initial conditions in the t-x plane.

8. page 77: 4b.

9. page 77: 7

10. Consider the IVP $x' = x^2, x(t_0) = x_0$

(a) For which initial points (t_0, x_0) are you assured that a unique solution exists?

(b) Find the solution for $t_0 = 0$, assuming $x_0 > 0$. At what value of t does the solution "blow up"? Plot $x(t)$ and state its interval of existence.

11. The stiffness k of a spring determines the elongation ΔL by which a spring extends if a mass m is attached. (See figure 2.2 in text.) Hooke's law states that

$$k\Delta L = mg \tag{1}$$

where g is the acceleration of a body in free fall due to gravity.

   (a) State the dimensions of k in terms of the base dimensions length (L), mass (M), time (T). Hint: use the fact that the equation (1) must be dimensionally correct.

   (b) If lengths are measured in cm, mass in grams, time in seconds (s) (that is, gravity in cm/s^2), what are the units of k?

12. When a mass of 0.3 kg is placed on a spring hanging at rest from the ceiling it elongates the spring by 5 cm.

   (a) What is the stiffness k of the spring? Make sure to include units and that its dimension agrees with your answer in 1(a).

   (b) Assume that there is no damping. Write an ordinary differential equation that governs the motion of the spring, in terms of the downward displacement $x(t)$ (as in Figure 2.2).

13. Suppose a spring in a medium with no damping satisfies

$$x'' + 25x = 0$$

Show that $x(t) = 2\cos(5t)$ solves the equation with $x(0) = 2$, $x'(0) = 0$. Plot $x(t)$.

14. Suppose a spring in a medium with some damping satisfies

$$x'' + 6x' + 25x = 0$$

Show that $x(t) = 2e^{-3t}\cos(4t)$ solves the equation with $x(0) = 2$, $x'(0) = -6$. Plot $x(t)$.

15. Suppose a spring in a medium with much more damping satisfies

$$x'' + 26x' + 25x = 0$$

Show that $x(t) = 4e^{-t} - e^{-25t}$ solves the equation with $x(0) = 3$, $x'(0) = 21$. Plot $x(t)$.

16. A spring satisfies

$$mx'' + \gamma x' + kx = 0 \quad (*)$$

where $m, k > 0$, $\gamma \geq 0$. The spring velocity is $v = x'$. Define the total energy of the system at time t to be

$$E(t) = \frac{m}{2}v^2 + \frac{k}{2}x^2 = \frac{m}{2}(x')^2 + \frac{k}{2}x^2$$

(a) Show that if $\gamma > 0$, the energy decreases. (Hint: show that $E'(t) < 0$. Use the chain rule and the equation (*).)

(b) Show that if $\gamma = 0$, the energy is conserved. (Show that $E'(t) = 0$. Use your result above.)