MATH 316 — REVIEW for Exam 4

Topics and sample problems, HW9-10

Vector-valued functions.

1. Be able to plot simple examples.
2. Check whether a given function solves a linear system of ODEs.

Linear Systems. You should be comfortable, in any of the cases (real distinct, complex, repeated eigenvalues),

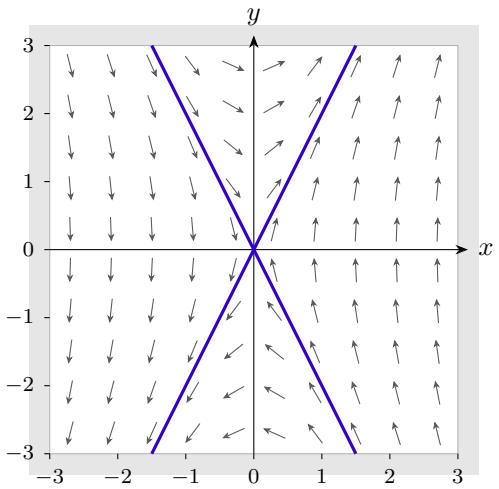
1. finding all equilibria
2. finding the general solution
3. solving an initial value problem (IVP)
4. plotting the phase portrait
5. relate the (unique) solution to an IVP to a curve in the phase portrait
6. know what the direction field is of a system and its relation to the phase portrait

All homework problems in HW 9–11 are good examples. You should be ready to redo any of those on the exam. Some additional sample problems are:

1. Find the inverse of the matrix $A = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$ and use it to solve the system of linear equations
$$\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}.$$
2. Write the second-order differential equation $x'' + 3x' + 9x = 0$ as a system of two first-order equations.
3. Show that $\mathbf{x}(t) = e^{2t} \begin{pmatrix} 4 \\ 2 \end{pmatrix}$, $t \in (-\infty, \infty)$, solves $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x}$ and sketch the graph of the (oriented) curve.
4. Show $\mathbf{x}(t) = \begin{pmatrix} 3\sin t \\ -3\cos t \end{pmatrix}$, $t \in (-\infty, \infty)$, solves $\mathbf{x}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{x}$ and sketch the graph of the (oriented) curve.
5. Find general solution, classify equilibrium, plot phaseportrait, solve IVP, plot the solution of the IVP in the phaseplane, for each of the following. If you are only asked to plot the phaseportrait, what information do you need?
   (a) $\mathbf{x}' = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$
   (b) $\mathbf{x}' = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$
   (c) $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$
   (d) $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$
   (e) $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

(f) $\mathbf{x}' = \begin{pmatrix} 2 & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

6. The differential equation $\mathbf{x}' = A\mathbf{x}$ has two straight line solutions corresponding to eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ that are shown on the direction field below. We denote the associated eigenvalues by λ_1 and λ_2.



What can you deduce about the eigenvalues λ_1 and λ_2?

7. The eigenvalues and eigenvectors for the coefficient matrix A in the linear homogeneous system $\mathbf{x}' = A\mathbf{x}$ are $\lambda_1 = 4$ with $v_1 = \langle 1, 2 \rangle$ and $\lambda_2 = -3$ with $v_2 = \langle -2, 1 \rangle$. What is the form of the solution? Plot the phase portrait.

8. The solution of a certain linear system $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

   (a) Which term dominates as $t \to \infty$?
   (b) Which term dominates as $t \to -\infty$?
   (c) Sketch a phaseportrait for this system.

9. For each of the following systems $\mathbf{x}' = A\mathbf{x}$,

   (i) Determine the eigenvalues of A and describe their dependence on α.
   (ii) Describe the dependence of the phase portrait on α.

   (a) $\mathbf{x}' = \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix} \mathbf{x}$

   (b) $\mathbf{x}' = \begin{pmatrix} 0 & -5 \\ 1 & \alpha \end{pmatrix} \mathbf{x}$

   (c) $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ \alpha & -2 \end{pmatrix} \mathbf{x}$

   (d) $\mathbf{x}' = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \alpha & \frac{5}{4} \end{pmatrix} \mathbf{x}$